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Situation N: Verification of an identity

Prompt

The most familiar settings for "verifying an identity" are in an Algebra class or in a Trigonometry class. But such tasks really pop up in almost every mathematics course at every level. The significant issues are the relevant background content and selecting a valid approach for the verification.

Commentary

The "verification" situation is universal. Assuming that the relevant mathematical skills are known, the teacher or student may still encounter trouble in recognizing the situation and then finding and carrying out a logical sequence of steps necessary to prove the identity true.

Some examples:

In Algebra, prove that |zw| = |z| |w| for complex numbers.

In Algebra, prove that $(AB)^{-1} = B^{-1}A^{-1}$ for 2 x 2 matrices.

In Trigonometry, prove that $\cos x[\sin x \tan x + 1] = 1 + \frac{1-\cos x}{\sec x}$

In Pre-calculus, many mathematical induction proofs boil down to verifying an identity.

In Calculus, verify that $\frac{d}{dx}\sqrt{5\ x^4\ + 6x - 7} = \frac{10x^3 + 3}{\sqrt{5\ x^4\ + 6x - 7}}$.

In Geometry, verify that $a = 2mn,\ b = m^2 - n^2,\ c = m^2 + n^2$ produce a right triangle.

Mathematical Foci

Mathematical Focus 1

Some possible approaches are

(1) work with one side and turn it into the other side

(2) work with each side separately and show that they reach a common result

(3) convert the proposed identity to an equivalent identity; repeat this step until a true statement is reached (many textbooks frown upon this approach).

(4) begin with a known truth and manipulate it until reaching the desired identity.

Mathematical Focus 2

There are often implicit constraints, which may or may not be understood.

For instance, consider the "identity" $\frac{x^2-1}{x-1} = x+1$. This is valid for all values of x except x = 1. Having recognized this limitation on x, the following sequence of equivalences is valid

$$\frac{x^2-1}{x-1} = x+1$$

$$\Leftrightarrow x^2 - 1 = (x-1)(x+1)$$

$$\Leftrightarrow x^2 - 1 = x^2 - 1.$$

The last statement is certainly true, so the equivalent first statement is true. But the reversibility and validity of the equivalence

$$\frac{x^2-1}{x-1}=x+1 \Leftrightarrow x^2-1=(x-1)(x+1)$$

depends upon division by the non-zero term x – 1.

Situation M: Algebraic simplifications involving exponents

Prompt

Naturally, exponents are introduced by examples such as 2^3 and $(-5)^2$. Even after students have mastered the concepts and some basic properties, more complicated situations are invariably much more difficult. Algebra is impossible without a mastery of exponents.

Commentary

The Rules of Exponents are fundamental in mathematics. But they seem to be the most difficult algebraic tool to understand and use. In particular, the appearance of rational and/or negative exponents in expressions in Algebra, Trigonometry and Calculus seems to be a perpetual cause for errors on the part of most students. Simply substituting into an expression or simplifying an algebraic expression or manipulating a derivative are prime sources of error.

Mathematical Foci

Mathematical Focus 1

A student must make the connection between a radical and a fractional exponent, such as $\sqrt{10} = 10^{1/2}$. the Rules of Exponents must be realized when manipulating: $16^{3/2} = \left(16^{1/2}\right)^3 = 4^3 = 64$ is preferable to $16^{3/2} = \left(16^3\right)^{1/2} = ugh$.

Mathematical Focus 2

The most fundamental tool in algebraic simplifications is "factoring out" a common factor. (It's nice if a student realizes that the distributive property is being used in reverse, but it is essential that he/she be able to carry out the steps.) When "powers" are involved, the student should learn lessons with positive integer exponents, then extend to "ugly powers:

(i) $5a^3 + 2a^2 = a^2(5a+2)$; "factor out the smaller power"

(ii) $5a^{-3} + 2a^2 = a^{-3}\left(5a + 2a^{2-(-3)}\right) = a^{-3}\left(5 + 2a^5\right)$;

(iii) $$3x\sqrt{4x^2+5} - 8\left(4x^2+5\right)^{3/2} = 3x\left(4x^2+5\right)^{1/2} - 8\left(4x^2+5\right)^{3/2}$$
$$= \left(4x^2+5\right)^{1/2}\left[3x - 8\left(4x^2+5\right)^{3/2-1/2}\right]$$
$$= \left(4x^2+5\right)^{1/2}\left[3x - 8\left(4x^2+5\right)^{1}\right]$$
$$= \left(4x^2+5\right)^{1/2}\left[3x - 32x^2 - 40\right].$$